

Critical and tricritical exponents of the Gross-Neveu model in the large- N_f limit

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Abstract

The critical and the tricritical exponents of the Gross-Neveu model are calculated in the large- N_f limit. Our results indicate that these exponents are given by the mean-field values.

1 Introduction

The thermodynamical properties of QCD is relatively unexplored subject which is nevertheless important in its own right [1]. Furthermore, it is currently receiving renewed interests because of the heavy ion collision experiments started at Relativistic Heavy Ion Collider (RHIC) which is expected to have potential of probing into transitions to the quark-gluon plasma phase.

It is, however, highly nontrivial problem how to extract the thermodynamic quantities of QCD from these experiments. The critical exponents, for example, are defined in equilibrium statistical mechanics and it is not obvious how they can be revealed in such energetic collision experiments. Therefore, it may be of help if we have a model which has the similar thermodynamic properties as QCD and at the same time simple enough to be tractable analytically.

It was shown in their original work that the Gross-Neveu model has the same physical properties as QCD, asymptotic freedom, dimensional transmutation, and the spontaneous chiral symmetry breaking [2]. The thermodynamics of the Gross-Neveu model in the large- N_f limit is discussed by Wolff who worked out the structure of the phase diagram of chiral symmetry breaking on chemical potential-temperature plane, and in particular found a tricritical point [3]. Furthermore, Chodos et al. have argued recently that by adding a pairing interactions the extended model admits a superconducting phase and possesses a very similar phase diagram with that of two-flavor massless QCD with chiral symmetry breaking and Cooper pairing phases [4]. Therefore, the model is a good candidate for the theoretical laboratory of QCD for studies its thermodynamic properties.

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In this paper, we study the thermodynamics of the original Gross-Neveu model and calculate the critical and tricritical exponents. It can be regarded as a preparatory study toward the discussions of how to extract thermodynamic quantities from reliable dynamical processes in nonequilibrium or moderately equilibrium environments. A work has already been initiated aiming at pursuing such a goal [5].

In this paper, we restrict ourselves to the large- N_f limit of the Gross-Neveu model. Since it is a $(1+1)$ -dimensional field theory, nontrivial phase structure at nonzero temperature is only possible in the large- N_f limit due to the Mermin-Wagner theorem [6]. The nature of the phenomenon and its interpretation as an "almost spontaneous symmetry breaking" has been discussed by Witten [7]. We think it sufficiently illuminative of the fact that the Gross-Neveu model in the large- N_f limit provides a consistent theory which is usable as a theoretical laboratory for QCD.

In Sec. 2 we review the Gross-Neveu model at finite chemical potential and temperature. In Secs. 3 and 4 we calculate the critical and the tricritical exponents of the Gross-Neveu model. The final section is devoted to conclusions.

2 The Gross-Neveu model at finite density and temperature

The Gross-Neveu model is a $(1+1)$ -dimensional model whose Lagrangian density at finite fermion density may be described by

$$\mathcal{L}_{\text{GN}}[\bar{\psi}, \psi; \mu, m_0] \equiv \bar{\psi}_i(i\partial - m_0)\psi_i + \frac{\lambda}{2N_f}(\bar{\psi}_i\psi_i)^2 + \mu\bar{\psi}_i\gamma_0\psi_i \quad (2.1)$$

where μ denotes the chemical potential and m_0 the bare mass. The index i denotes flavor of fermions and runs from 1 to N_f . Note that when we set $m_0 = 0$ the model has an invariance under the discrete chiral transformation $\bar{\psi} \rightarrow -\bar{\psi}\gamma_5$, $\psi \rightarrow \gamma_5\psi$. The theory is renormalizable, by virtue of that the model is $(1+1)$ -dimensional one.

To make the model easy to handle we introduce an auxiliary field σ by adding a term

$$\mathcal{L}_{\text{aux}} \equiv -\frac{N_f}{2\lambda} \left(\sigma - m_0 + \frac{\lambda}{N_f} \bar{\psi}_i\psi_i \right)^2 \quad (2.2)$$

to the original Lagrangian density \mathcal{L}_{GN} . Considering the fact that $\int \mathcal{D}\sigma \exp(i \int d^2x \mathcal{L}_{\text{aux}})$ is an irrelevant constant we can use the Lagrangian density

$$\mathcal{L} \equiv \mathcal{L}_{\text{GN}} + \mathcal{L}_{\text{aux}} = \bar{\psi}_i(i\partial + \mu\gamma_0 - \sigma)\psi_i - \frac{N_f}{2\lambda}(\sigma - m_0)^2 \quad (2.3)$$

without changing the dynamics of the original Lagrangian density.

For uniform vacuum we define the effective potential $V_{\text{eff}}(\sigma)$ as $\Gamma_{\text{eff}} = -iN_f(\int d^2x) V_{\text{eff}}$, where Γ_{eff} is the effective action. Now we take N_f infinity. In the limit we can construct V_{eff} from tree and one fermion loop diagrams as shown in Fig. 1. We obtain

$$V_{\text{eff}}(\sigma, \mu, m_0) = \frac{1}{2\lambda}(\sigma - m_0)^2 + i \int \frac{d^2k}{(2\pi)^2} \ln \left(-(k_0 + \mu)^2 + k_1^2 + \sigma^2 \right) \quad (2.4)$$

up to a constant term. From the equation of motion for σ we see

$$\sigma_{\text{cl}} \equiv \langle \sigma \rangle = m_0 - \frac{\lambda}{N_f} \langle \bar{\psi}_i(x) \psi_i(x) \rangle. \quad (2.5)$$

The vacuum expectation value σ_{cl} is the physical mass of fermions and also order parameter for the discrete chiral symmetry breaking.

So far we dealt with zero temperature case, but from now let us put the system into thermal bath of finite temperature T . That is achieved by replacing k_0 -integral with sum over Matsubara frequencies as follows:

$$\int \frac{dk_0}{2\pi} \rightarrow \sum_{n=-\infty}^{\infty} iT, \quad k_0 \rightarrow (2n+1)\pi iT \quad (2.6)$$

where the Boltzmann constant is set to be unity. After the replacements the effective potential becomes

$$V_{\text{eff}}(\sigma, \mu, T, m_0) = \frac{1}{2\lambda}(\sigma - m_0)^2 + i \int \frac{dk_1}{2\pi} \sum_{n=-\infty}^{\infty} iT \ln \left[-\left\{ (2n+1)\pi iT + \mu \right\}^2 + k_1^2 + \sigma^2 \right]. \quad (2.7)$$

Now we must do renormalization, because the second term of (2.7) clearly diverges. Naively speaking, $1/\lambda$ and m_0 are the parameters which we can use for renormalization. But it is better to regard $1/\lambda$ and m_0/λ as independent parameters. We define a renormalized coupling constant λ_R as follows:

$$\begin{aligned} \frac{1}{\lambda_R} &\equiv \left. \frac{\partial^2 V_{\text{eff}}}{\partial \sigma^2} \right|_{\substack{\sigma=\sigma' \\ \mu=T=0}} = \frac{1}{\lambda} + \frac{1}{\pi} - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dk_1 \frac{k_1^2}{\{k_1^2 + (\sigma')^2\}^{3/2}} \\ &= \frac{1}{\lambda} + \frac{1}{\pi} - \frac{1}{2\pi} \ln \left(\frac{2\Lambda}{\sigma'} \right)^2 \end{aligned} \quad (2.8)$$

where σ' denotes an arbitrary renormalization point and Λ is a momentum cutoff. In fact the divergence in (2.7) can be absorbed by only the coupling constant renormalization. Therefore, m_0/λ is finite and needs not to be renormalized. Now it is convenient to define m_F by

$$m_F \equiv \sigma' \exp \left(1 - \frac{\pi}{\lambda_R} \right) \quad (2.9)$$

which represents the dynamically generated mass of fermions for $\mu = T = m_0 = 0$. We obtain the gap equation

$$\begin{aligned} 0 &= \left. \frac{\partial V_{\text{eff}}(\sigma, \mu, T, m_0)}{\partial \sigma} \right|_{\sigma=\sigma_{\text{cl}}} \\ &= \frac{\sigma_{\text{cl}}}{2\pi} \ln \frac{\pi^2 T^2}{m_F^2} - \frac{m_0}{\lambda} - \frac{\gamma_E \sigma_{\text{cl}}}{\pi} \\ &\quad - \sum_{n=0}^{\infty} \text{Re} \frac{2\sigma_{\text{cl}}}{\left[\{(2n+1)\pi + i\mu/T\}^2 + \sigma_{\text{cl}}^2/T^2 \right]^{1/2}} + \sum_{n=0}^{\infty} \frac{2\sigma_{\text{cl}}}{(2n+1)\pi} \end{aligned} \quad (2.10)$$

where γ_E denotes Euler's constant ($\simeq 0.5772$), see [8] for detailed calculation. The phase diagram on μ - T plane for $m_0 = 0$ was obtained in [2] (Fig. 2). In the region surrounded by the curve $ABCO$ the discrete chiral symmetry is spontaneously broken. On the curve AB there occurs a second-order phase transition, while BD is the first-order phase transition line. Thus, the Gross-Neveu model in the large- N_f limit has a tricritical point B .

3 The critical exponents of the Gross-Neveu model

We study the thermodynamical properties of the Gross-Neveu model in the large- N_f limit. At first we obtain the critical exponents analytically. In order to do this we set $\mu = 0$ for the time being. Then the gap equation (2.10) becomes

$$\begin{aligned}
0 &= \left. \frac{\partial V_{\text{eff}}(\sigma, \mu = 0, T, m_0)}{\partial \sigma} \right|_{\sigma=\sigma_{\text{cl}}} \\
&= \frac{\sigma_{\text{cl}}}{2\pi} \ln \frac{\pi^2 T^2}{m_F^2} - \frac{m_0}{\lambda} - \frac{\gamma_E \sigma_{\text{cl}}}{\pi} \\
&\quad - \sum_{n=0}^{\infty} \frac{2\sigma_{\text{cl}}}{[\{(2n+1)\pi\}^2 + \sigma_{\text{cl}}^2/T^2]^{1/2}} + \sum_{n=0}^{\infty} \frac{2\sigma_{\text{cl}}}{(2n+1)\pi} \\
&= \frac{\sigma_{\text{cl}}}{2\pi} \ln \frac{\pi^2 T^2}{m_F^2} - \frac{m_0}{\lambda} - \frac{\gamma_E \sigma_{\text{cl}}}{\pi} + \frac{7\zeta(3)\sigma_{\text{cl}}^3}{8\pi^3 T^2} + o(\sigma_{\text{cl}}^5 T^{-4}). \tag{3.1}
\end{aligned}$$

In the last equality we have used high temperature expansion [9] which can be used for the region $1/T < \pi/|\sigma_{\text{cl}}|$. We can use the expansion near the second-order phase transition point, because σ_{cl} is almost zero and T is nonzero. The critical temperature T_c (the temperature of the point A in Fig. 2) has been given [10] by solving

$$0 = \left. \frac{1}{\sigma} \frac{\partial V_{\text{eff}}}{\partial \sigma} \right|_{\substack{\sigma=\mu=m_0=0 \\ T=T_c}} = \frac{1}{2\pi} \ln \frac{\pi^2 T_c^2}{m_F^2} - \frac{\gamma_E}{\pi}, \tag{3.2}$$

and the result is

$$T_c = \frac{m_F}{\pi} e^{\gamma_E} \simeq 0.566933 m_F. \tag{3.3}$$

When the temperature is T_c and m_0 goes to zero, the gap equation becomes

$$0 = \left. \frac{\partial V_{\text{eff}}}{\partial \sigma} \right|_{\substack{\sigma=\sigma_{\text{cl}} \\ \mu=0 \\ T=T_c}} = -\frac{m_0}{\lambda} + \frac{7\zeta(3)}{8\pi^3 T_c^2} \sigma_{\text{cl}}^3 + o(\sigma_{\text{cl}}^5 T_c^{-4}) \tag{3.4}$$

and the behavior of the order parameter σ_{cl} is given by

$$\sigma_{\text{cl}} \simeq m_0^{1/3} \equiv m_0^{1/\delta}. \tag{3.5}$$

Therefore, we obtain a critical exponent $\delta = 3$. On the other hand, when $m_0 = 0$ and the temperature closes to T_c from bellow, the gap equation becomes

$$0 = \left. \frac{\partial V_{\text{eff}}}{\partial \sigma} \right|_{\substack{\sigma = \sigma_{\text{cl}} \\ \mu = m_0 = 0}} = \frac{\sigma_{\text{cl}}}{2\pi} \ln \frac{T^2}{T_c^2} + \frac{7\zeta(3)}{8\pi^3 T^2} \sigma_{\text{cl}}^3 + o(\sigma_{\text{cl}}^5 T_c^{-4})$$

$$\stackrel{T \simeq T_c}{\simeq} -\frac{T_c^2 - T^2}{2\pi T_c^2} \sigma_{\text{cl}} + \frac{7\zeta(3)}{8\pi^3 T^2} \sigma_{\text{cl}}^3 + o(\sigma_{\text{cl}}^5 T_c^{-4}) \quad (3.6)$$

and the behavior of σ_{cl} is

$$\sigma_{\text{cl}} \simeq (T_c - T)^{1/2} \equiv (T_c - T)^\beta. \quad (3.7)$$

Thus, we obtain a critical exponent $\beta = 1/2$. There exist other critical exponents α and γ defined by

$$C_V|_{m_0=0} \equiv \frac{\partial^2}{\partial T^2} V_{\text{eff}}|_{\substack{\sigma = \sigma_{\text{cl}}(T) \\ m_0 = 0}} \simeq (T_c - T)^{-\alpha}, \quad \chi|_{m_0=0} \equiv \frac{\partial \sigma_{\text{cl}}}{\partial m_0} \Big|_{m_0=0} \simeq (T_c - T)^{-\gamma}. \quad (3.8)$$

which govern the behavior of the specific heat C_V and the susceptibility χ near the critical point. Because we have already obtained two exponents $\delta = 3$ and $\beta = 1/2$, we can obtain them by using the scaling relations

$$\alpha = 2 - \beta(\delta + 1) = 0, \quad \gamma = \beta(\delta - 1) = 1. \quad (3.9)$$

4 The tricritical exponents of the Gross-Neveu model

From now on we reinstall the chemical potential μ and study the tricritical exponents of the Gross-Neveu model in the large- N_f limit. See, for example, [11] for an extensive review of the tricritical phenomena. We solve the gap equation (2.10) numerically and see the behavior of the solution σ_{cl} near the tricritical point. First of all we need to know the tricritical point μ_{tc} and T_{tc} . They are obtained by numerically solving the equations

$$0 = \left. \frac{\partial^2 V_{\text{eff}}}{\partial \sigma^2} \right|_{\sigma=m_0=0} = \lim_{n_{\text{max}} \rightarrow \infty} \left\{ \frac{1}{2\pi} \ln \frac{16\pi^2 n_{\text{max}}^2 T^2}{m_F^2} - 2 \sum_{n=0}^{n_{\text{max}}} \text{Re} \frac{1}{(2n+1)\pi + i\mu/T} \right\}, \quad (4.1)$$

$$0 = \left. \frac{\partial^4 V_{\text{eff}}}{\partial \sigma^4} \right|_{\sigma=m_0=0} = \frac{2}{T^2} \sum_{n=0}^{\infty} \text{Re} \frac{1}{\{(2n+1)\pi + i\mu/T\}^3} \quad (4.2)$$

and the results are given [3] by

$$\mu_{\text{tc}} \simeq 0.608221 m_F, \quad T_{\text{tc}} \simeq 0.318328 m_F. \quad (4.3)$$

We can achieve high temperature expansion of (2.10)[8] which can be used for the region $1/T < \pi/(|\sigma_{\text{cl}}| + |\mu|)$. Although the condition is satisfied near the tricritical point, the expansion does not enable us to calculate the tricritical exponents analytically because the

power series of $1/T$ is not of σ for $\mu \neq 0$. Therefore, we calculate the tricritical exponents numerically.

At the tricritical point the behavior of σ_{cl} will be characterized by a tricritical exponent δ_{tc} as $\sigma_{\text{cl}} \simeq m_0^{1/\delta_{\text{tc}}}$ for sufficiently small m_0 . Our numerical computation indicates

$$\frac{1}{\delta_{\text{tc}}} \simeq \left. \frac{m_0}{\sigma_{\text{cl}}} \frac{\partial \sigma_{\text{cl}}}{\partial m_0} \right|_{\substack{\mu = \mu_{\text{tc}} \\ T = T_{\text{tc}} \\ m_0 \simeq 0}} \simeq 0.20 \quad (4.4)$$

as shown in Fig. 3. We roughly estimate the error of $1/\delta_{\text{tc}}$ to be less than 0.01.

Next we set $m_0 = 0$ and investigate the behavior of σ_{cl} near the tricritical point. Though there are many paths to approach to the tricritical point and be two exponents which relate with the paths, at first we choose simply the path of $T = T_{\text{tc}}$. As approach to the tricritical point along the path, the order parameter will behave as $\sigma_{\text{cl}} \simeq (\mu_{\text{cl}} - \mu)^{\beta_l}$ with a tricritical exponent β_l . Our numerical computation indicates

$$\beta_l \simeq \left. \frac{\mu_{\text{tc}} - \mu}{\sigma_{\text{cl}}} \frac{\partial \sigma_{\text{cl}}}{\partial (\mu_{\text{tc}} - \mu)} \right|_{\substack{\mu \simeq \mu_{\text{tc}} \\ T = T_{\text{tc}} \\ m_0 = 0}} \simeq 0.25 \quad (4.5)$$

and is shown in Fig. 4. We roughly estimate the error of β_l to be less than 0.01.

There should be another exponent corresponding other paths, but to obtain it is rather difficult. For example the path of $\mu = \mu_{\text{tc}}$ gives just same exponent as β_l and in fact any linear paths do also. Here remember that the curve AB in Fig. 2 causes the second-order phase transition and that the curve AB and BC are parts of the solution of an equation (4.1). The path of $T = T_{\text{tc}}$ which crosses the curve ABC had given an tricritical exponent $\beta_l \simeq 0.25$, therefore the path along BC (in broken phase) may give another exponent which must exist. For the reason we define the parameter s which parametrizes the curve BC in units of m_{F} so that $s = 0$ at the tricritical point B . Then, the order parameter σ_{cl} behaves as $\sigma_{\text{cl}} \simeq s^{\beta_t}$ with a tricritical exponent β_t near the tricritical point. Our numerical computation indicates

$$\beta_t \simeq \left. \frac{s}{\sigma_{\text{cl}}} \frac{\partial \sigma_{\text{cl}}}{\partial s} \right|_{\substack{s \sim 0 \\ m_0 = 0}} \simeq 0.5 \quad (4.6)$$

as shown in Fig. 5. That is indeed the other exponent than β_l that we have looked for. But the accuracy of β_l is less than that for β_t because of rather intricate way of approaching to the tricritical point. Though there are more tricritical exponents α_l and α_t for the specific heat, γ_l and γ_t for the susceptibility, we can obtain them from the scaling relations

$$\begin{aligned} \alpha_l &= 2 - \beta_l(\delta_{\text{tc}} + 1) \simeq 1.5, & \gamma_l &= \beta_l(\delta_{\text{tc}} - 1) \simeq 1, \\ \alpha_t &= 2 - \beta_t(\delta_{\text{tc}} + 1) \simeq -1, & \gamma_t &= \beta_t(\delta_{\text{tc}} - 1) \simeq 2. \end{aligned} \quad (4.7)$$

It is remarkable that α_t has a negative value. It indicates that the specific heat is continuous at the tricritical point as a function of s , which is different from what is expected in usual second-order phase transition.

Finally, let us confirm the validity of our numerical calculations for the tricritical exponents. By essentially the same computer program for the tricritical exponents, we numerically calculate the critical exponents δ and β which have been already obtained analytically. The results are shown in Fig. 6 and Fig. 7, and these results agree with analytical ones very well. Therefore, we believe that our numerical calculations for the tricritical exponents are also accurate enough.

5 Conclusion

In this paper, we have calculated the critical exponents of the Gross-Neveu model analytically and the tricritical exponents numerically in the large- N_f limit. The results we have obtained are as follows:

$$\text{the critical exponents} \quad : \quad \delta = 3, \beta = \frac{1}{2}$$

$$\text{the tricritical exponents} \quad : \quad \frac{1}{\delta_{tc}} = 0.20, \beta_l = 0.25, \beta_t = 0.5.$$

The accuracy of our numerical calculations of the tricritical exponents are checked against the values of the critical exponents which we can obtain analytically. The errors in the tricritical exponents are thereby estimated as less than a few %.

Our results indicate that all the critical and tricritical exponents of the Gross-Neveu model in the large- N_f limit are given by the mean-field values [12]. The reason may be that the fluctuations around a mean field are suppressed in the leading order of $1/N_f$ expansion. Unfortunately, to calculate the next order of $1/N_f$ does not help because the critical temperature becomes exactly zero at finite N_f in the Gross-Neveu model; it is believed that the result of [6] applies to any field theories in $(1+1)$ -dimensions with short range interactions.

Then, what is the significance of the computation of critical exponents we have carried out in this paper? It is to prepare for the study of the question of how equilibrium thermodynamical quantities can be signaled and extracted from the experiments, as we have discussed in Introduction. However, we have to keep in mind, despite their similarities, the following differences between the Gross-Neveu model and QCD. Namely, we have no reason to expect the same numerical values of the critical and the tricritical exponents, and furthermore their physical nature may not be the same. It is because the Gross-Neveu model in the large- N_f limit have the mean-field critical exponents, while QCD with two massless flavor is believed to belong to the universality class of $O(4)$ spin model [13]. We hope that these differences do not seriously disturb the validity of the Gross-Neveu model as a theoretical laboratory for studying thermodynamic properties of QCD.

6 Acknowledgment

I wish to thank Prof. H. Minakata for suggesting the problem and his help, and all the members of our laboratory for useful discussions.

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$$-iN_f V_{\text{eff}} = \bullet \cdots \sigma + \left[\text{circle with two arrows} - \frac{1}{[\text{dashed line}]} \right] \frac{\sigma^2}{2} + \text{circle with four arrows} \frac{\sigma^4}{4!} + \dots$$

Figure 1: The diagrams which contribute to the effective potential in large- N_f limit. Only the terms up to fourth order in σ are explicitly exhibited.

Figure 2: The phase diagram of the Gross-Neveu model in the large- N_f limit and the forms of the effective potential each for $m_0 = 0$ [3]. The point B is the tricritical point.

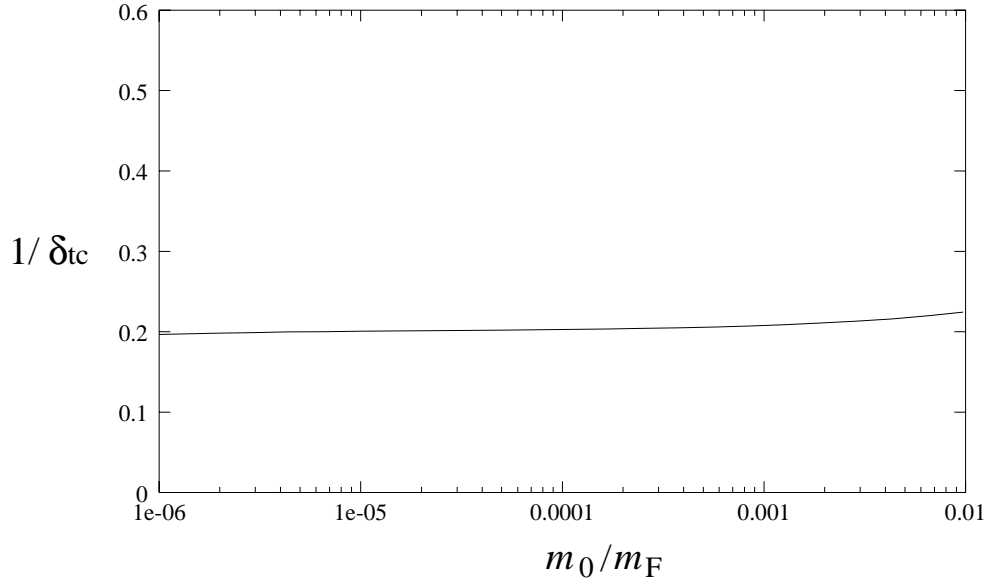


Figure 3: The tricritical exponent $1/\delta_{tc}$ is plotted as a function of m_0/m_F . Here, μ and T are set to be μ_{tc} and T_{tc} respectively. In the vanishing m_0 limit $1/\delta_{tc}$ approaches to 0.2.

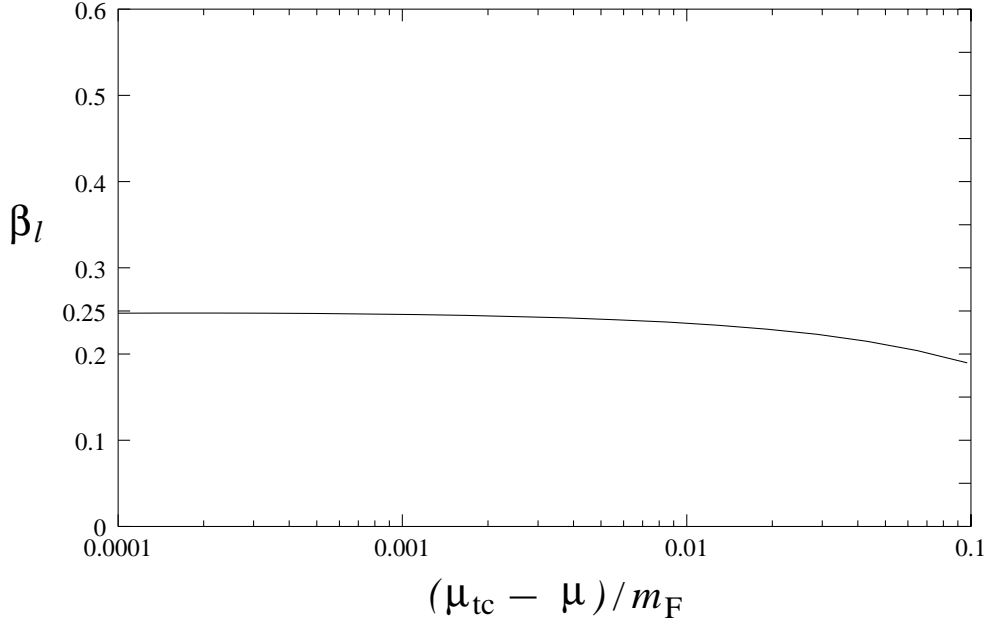


Figure 4: The tricritical exponent β_l is plotted as a function of $(\mu_{tc} - \mu)/m_F$. Here, $m_0 = 0$ and T is set to be T_{tc} . As μ closes to μ_{tc} , β_l approaches to 0.25.

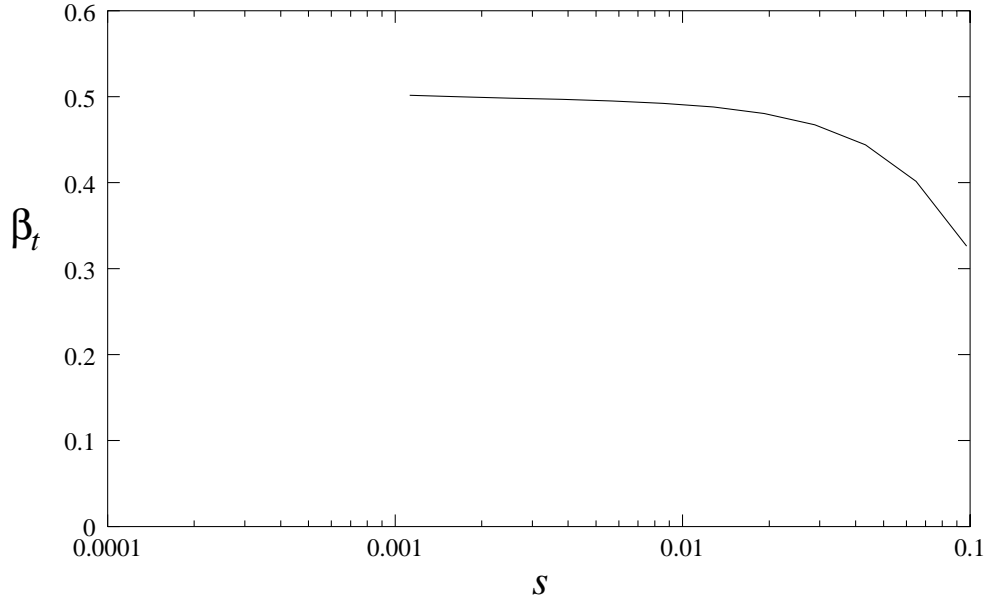


Figure 5: The tricritical exponent β_t is plotted as a function of s which parametrizes the curve BC in Fig. 2 in units of m_F so that $s = 0$ at the tricritical point B . The unit of s is m_F . Here, m_0 is set to be zero. As s approaches to the tricritical point, β_t approaches to 0.5.

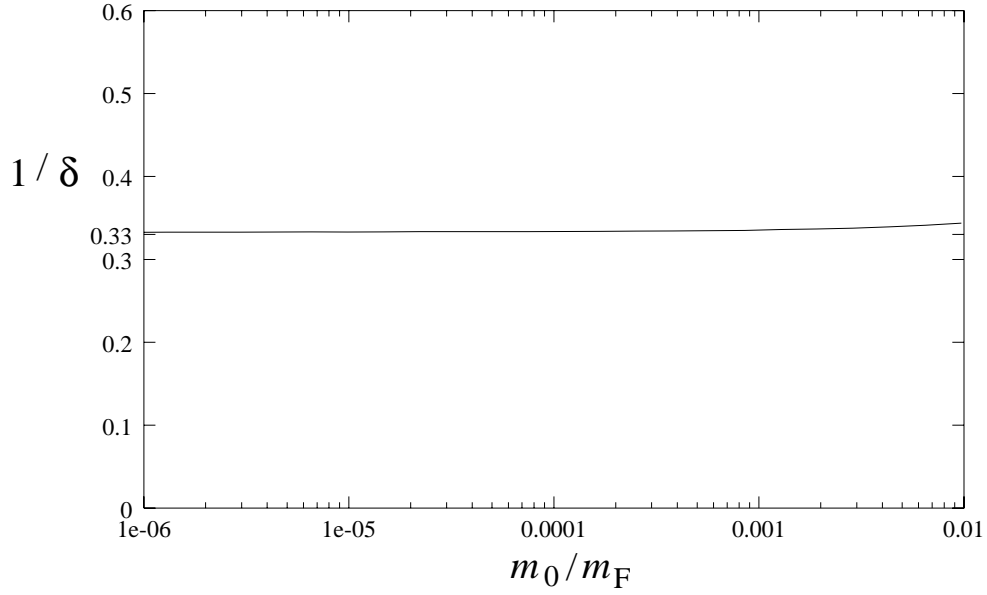


Figure 6: The critical exponent $1/\delta$ is plotted numerically as a function of m_0/m_F . Here, $\mu = 0$ and T is set to be T_c . In the vanishing m_0 limit $1/\delta$ approaches to 0.33 which is contrasted with the analytical result $1/\delta = 1/3$.

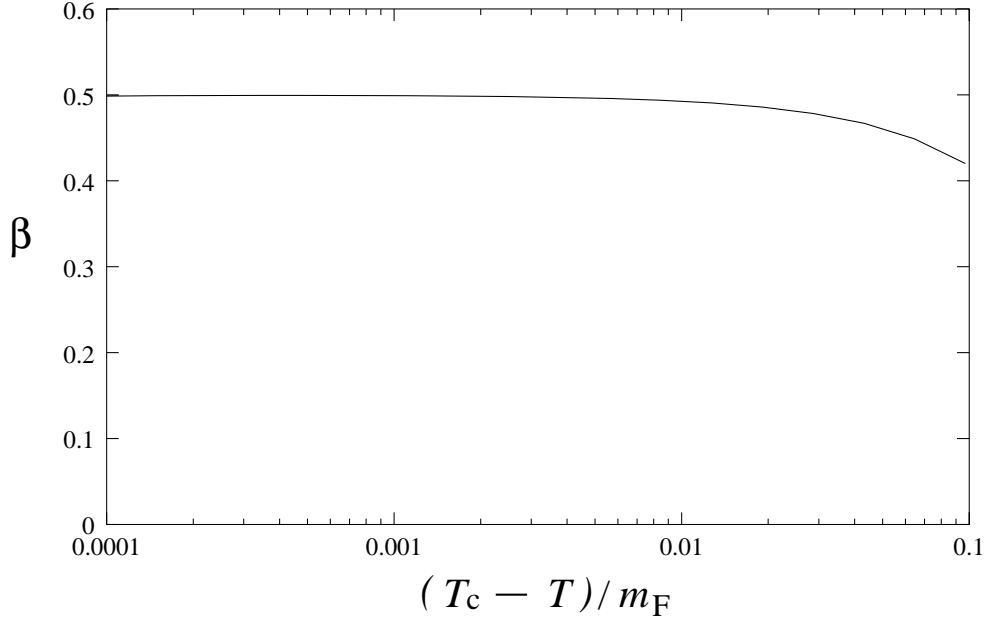


Figure 7: The critical exponent β is plotted numerically as a function of $(T_c - T)/m_F$. Here, m_0 and μ are set to be zero. As T closes to T_c , β approaches to 0.5 which is contrasted with the analytical result $\beta = 1/2$.